What Holfeld could not see in his solution

Ioannis Holfeld, *Exercitationes geometricae,* Prague, 1773, p. 27 – 28.

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Solutio: Sit $AN = x$; $NQ = y$; $AM = a$; $MB = b$; erit: $x : y = a : b + x$. Hinc $ay = bx + x^2$; quæ eſt æquatio ad Parabolam. Eſt enim $ay + \frac{1}{4}b^2 = x^2 + bx + \frac{1}{4}b^2 = (x + \frac{1}{2}b)^2$. Fiat $x + \frac{1}{2}b = z$; erit $ay + \frac{1}{4}b^2 = z^2$. Ad a, & $\frac{1}{2}b$, quære tertium proportionalem terminum; qui ſit m; eritque $am = \frac{1}{4}b^2$; &: $ay + am = z^2$. Sit $y + m = u$; erit $au = z^2$; quæ eſt æquatio ſimpliciſſima ad Parabolam.

PROBLEMA 24.

32. Datis duabus rectis AM, MB, productaque ad libitum MB in O, ſumatur a puncto A recta AR æqualis MO; quæritur locus interſectionum rectarum RB, AO.

Solutio: Sit $AN = x$; $NQ = y$; $AM = a$; $MB = b$; erit $x : y = a : b + BO$; igitur: $BO = \frac{ay - bx}{x}$. Hinc: $RA = b + \frac{ay - bx}{x} = \frac{ay}{x}$; & $RN = \frac{ay + x^2}{x}$; $RM = \frac{ay + ax}{x}$. Eſt autem: $\frac{ay + x^2}{x} : y = \frac{ay + ax}{x} : b$. Nam: $RN : NQ = RM : MB$. Igitur: $y^2 + y(x - b) = \frac{bx^2}{a}$. Quæ æquatio ad ſimpliciſſimam ita reducitur: $y^2 + y(x - b) + \frac{(x-b)^2}{4} = \frac{(x-b)^2}{4} + \frac{bx^2}{a}$; Fiat $y + \frac{x - b}{2} = z$; erit $z^2 = \frac{(x-b)^2}{4} + \frac{bx^2}{a}$; & $4az^2 = ax^2 + ab^2 - 2abx + 4bx^2$; $4az^2 - ab^2 = x^2(a + 4b) - 2abx$.

Sive:

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Sive: $\frac{4az^2 - ab^2}{a + 4b} = x^2 - \frac{2abx}{a + 4b}$. Sit $\frac{ab}{a + 4b} = m$; erit: $\frac{4az^2 - ab^2}{a + 4b} = x^2 - 2mx$; & $\frac{4az^2 - ab^2}{a + 4b} + m^2 = x^2 - 2mx + m^2 = (x - m)^2$; Fiat: $x - m = u$; erit $u^2 = \frac{4az^2 - ab^2}{a + 4b} + m^2$; & $\frac{4az^2}{a + 4b} = u^2 + \frac{ab^2}{a + 4b} - \frac{a^2b^2}{(a + 4b)^2} = u^2 + \frac{4ab^3}{(a + 4b)^2}$; quæ eſt æquatio ad hyperbolam; cujus deſcriptio (facta ſemper $AR = MO$) abſolvitur ductu rectarum AO, RB.

PROBLEMA 25.

35. Datis AM, MB, angulum rectum comprehendentibus, & ſumpta quavis BO, ductaque AO, quæ ſecetur in Q ea lege, ut ſit $OQ : QA = OB : OM$, invenire locum punctorum Q.

Solutio: Sit rurſum $AN = x$, $NQ = y$, $AM = a$; $MB = b$; erit: $AQ = \sqrt{x^2 + y^2}$; BO autem eſt $= \frac{ay - bx}{x}$, ut ante; & $OM = \frac{ay}{x}$; $AO^2 = a^2 + \frac{a^2y^2}{x^2}$; $AO = \frac{a}{x}\sqrt{x^2 + y^2}$; & $QO = \frac{(a - x)}{x}\sqrt{x^2 + y^2}$. Eſt vero ex conditione Problematis: $\frac{(a - x)}{x}\sqrt{x^2 + y^2} : \sqrt{x^2 + y^2} = \frac{ay - bx}{x} : \frac{ay}{x}$; ſeu: $a - x : x = ay - bx : ay$; &

a:

Problem 24: Given two lines AM, MB; select any point O on the line MB, construct the point R on the line AM such that AR is equal to MO; asking for the set of all intersections of two lines RB, AO.

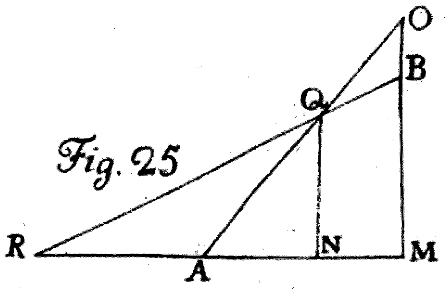

Figure 1: Illustration to *Problema 24*

Solution according to Holfeld:

Let $AN = x$; $NQ = y$; $AM = a$; $MB = b$. First, from the similarity of triangles ANQ a AMO follows the relation $x : y = a : (b + BO)$, i.e. $BO = \frac{ay - bx}{x}$. Applying the equality $RA = MO$ from the assignment we get $RA = b + BO = b + \frac{ay - bx}{x} = \frac{ay}{x}$. From the assignment also follows the equalities $RN = RA + x$, i.e. $RN = \frac{ay + x^2}{x}$, and $RM = RA + a = \frac{ay + ax}{x}$. Then, the similarity of triangles RNQ a RMB implies the relation $RN : y = RM : b$. Substituting for RN and RM we get the form $\frac{ay + x^2}{x} : y = \frac{ay + ax}{x} : b$ that can be simplified as follows $y^2 + y(x - b) = \frac{bx^2}{a}$. Finally, completing the square on the left side we get the equation of the studied locus (see Fig. 2):

$$\left(y + \frac{x - b}{2}\right)^2 = \frac{(x - b)^2}{4} + \frac{bx^2}{a}. \qquad (2)$$

Ioannis Holfeld proceeded with modifications of (2) to get the equation $\frac{4az^2}{a + 4b} = u^2 + \frac{4ab^3}{(a + 4b)^2}$ of a conic section in its reduced canonical form that he classified as a equation of hyperbola, which represents the set of intersections of lines *AO*, *RB*.

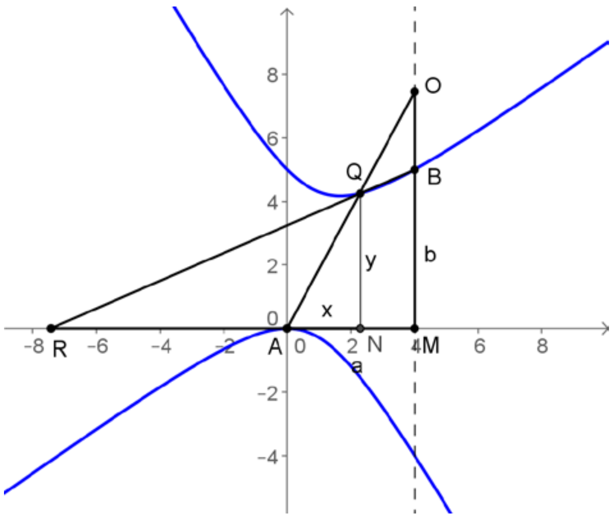

Figure 2: Graph of (2), $a = 4$, $b = 5$

TASK 1: Plot the equation (2) in GeoGebra as shown in Fig. 2. Use various values of *a* and *b*.

TASK 2: Create a dynamic model of “Problema 24”. Inspect the locus of point Q. Be aware of the fact that Ioannis Holfeld did not solve the problem completely. Will you be more successful?

TASK 3: Modify the method of Ioannis Holfeld to find the rest of the locus that GeoGebra hopefully reveals you within the solution to TASK 2.